Flow between torsionally oscillating disks

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Two parallel infinite plane disks, between which is contained a viscous fluid, oscillate torsionally about a common axis. Of specific interest are the cases where (i) one disk only is in motion, and (ii) the disks oscillate 180° out of phase, but with the same frequency and amplitude. The basic parameter is found to be the Reynolds number derived from the frequency, the kinematic viscosity and the disk separation. Solutions of the Navier–Stokes equations for both small and large Reynolds numbers are developed and the transverse and radial-axial flows investigated.

1. Introduction

In a recent paper (Rosenblat 1959) the author examined the flows resulting from the small torsional oscillations of an infinite disk in a viscous fluid otherwise unbounded and at rest. It was found that linearizing expansions depending on the smallness of the oscillation amplitude were insufficient to describe one of the velocity components, namely, the mean steady radial-axial component. This was essentially because the method does not allow for the fact that non-linear inertia terms predominate outside a thin boundary layer, though they are negligible within it.

As a sequel, the present paper investigates the analogous problem in which the fluid is now bounded by a second, parallel disk at a given distance. Two cases only are studied: (i) when one disk performs small torsional oscillations and the other is at rest; and (ii) when both disks oscillate with the same amplitude and frequency, though with a phase difference of 180°.

Commencing with the Navier–Stokes equations, we seek solutions on the assumption that certain non-linear inertia terms may be omitted. It is found that, unlike in the above-mentioned problem, such solutions are in fact obtainable with all boundary conditions satisfied. Moreover, it is subsequently shown that the approximate method is valid provided a single condition relating the amplitude of the oscillation with the Reynolds number of the flow holds.

Although exact solutions can be developed throughout they prove cumbersome, and approximations for low and high Reynolds number are more convenient. The former apply for $R < 10$ in case (i) and $R < 40$ in case (ii); the latter for $R > 20$ and $R > 80$, respectively.

Transverse velocities are first calculated and are seen to have a boundary-layer character at large R. The rotational motion of the disks gives rise to centrifugal forces which in turn cause the fluid to be thrown radially outwards. Hence a radial-axial flow is set up which has a mean steady component as well as a fluctuating component. The behaviour of the steady term is of particular interest and is fully discussed.

In order to balance the fluid expelled by centrifugal action a radial inflow between the disks is required. Hence a radial pressure gradient is induced which maintains this flow. It is found from the equations of motion to act inwards and to be constant. At high Reynolds numbers it is the dominant factor outside the boundary layers so that the flow there has a Poiseuille-type profile.

The case of large amplitude oscillations is not considered here. In the limit as the amplitude becomes infinite the problem becomes that studied by Batchelor (1951) and Stewartson (1953), where two parallel disks rotate steadily about a common axis. Some formal comparison of results may therefore be made.

2. One disk oscillating

Consider a body of fluid bounded by two parallel disks which are represented by the planes $z = 0$ and $z = d$ in a cylindrical polar co-ordinate system. The axis $r = 0$, perpendicular to the disks, is the axis about which they may perform torsional oscillations. If u, v and w be respectively the radial, transverse and axial velocity components, p the pressure, ρ the density and ν the kinematic viscosity, then the Navier–Stokes equations of motion are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{\partial^2 u}{\partial z^2}\right], \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+w\frac{\partial v}{\partial z}+\frac{uv}{r} = \nu\left[\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{\partial^2 v}{\partial z^2}\right], \tag{2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{\partial^2 w}{\partial z^2}\right], \tag{3}$$

while the equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{\partial w}{\partial z} = 0. \tag{4}$$

We take the disk $z = 0$ to perform torsional oscillations of frequency n and angular speed Ω, while the disk $z = d$ remains at rest. The boundary conditions are then

$$\left.\begin{aligned} &u = w = 0, \quad v = r\Omega e^{int} \quad \text{on} \quad z = 0,\\ &u = v = w = 0 \quad \text{on} \quad z = d. \end{aligned}\right\} \tag{5}$$

Equations (1)–(5) can be satisfied by writing

$$\left.\begin{aligned} &v = r\Omega e^{i\tau}g(y), \quad u = \frac{r\Omega^2}{n}\frac{\partial F}{\partial y}(y,\tau), \quad w = -2d\frac{\Omega^2}{n}F(y,\tau),\\ &p/\rho = \Omega^2 d^2 P(y,\tau)+\tfrac{1}{2}\Omega^2 r^2 K(\tau), \quad y = z/d, \quad \tau = nt. \end{aligned}\right\} \tag{6}$$

That is, we seek a solution in which the radial and transverse velocities are linear functions of r. The equations of motion then establish the required form of the pressure, with $P\ (y,\tau)$ and $K(\tau)$ unknown functions of the variables indicated.

On substitution from (6), equations (1) and (2) become

$$\frac{\partial^2 F}{\partial y\,\partial \tau}+\left(\frac{\Omega}{n}\right)^2\left[\left(\frac{\partial F}{\partial y}\right)^2-2F\frac{\partial^2 F}{\partial y^2}\right]-(g\,e^{i\tau})^2=-K(\tau)+\frac{1}{R}\frac{\partial^3 F}{\partial y^3} \tag{7}$$

and

$$ig+2\left(\frac{\Omega}{n}\right)^2\left[g\frac{\partial F}{\partial y}-F\frac{dg}{dy}\right]=\frac{1}{R}\frac{d^2 g}{dy^2}, \tag{8}$$

where

$$R\equiv nd^2/\nu \tag{9}$$

is the Reynolds number of the flow. The boundary conditions (5) become

$$\left.\begin{aligned} F=\frac{\partial F}{\partial y}=0,\quad g=1 \quad \text{on} \quad y=0,\\ F=\frac{\partial F}{\partial y}=g=0 \quad \text{on} \quad y=1. \end{aligned}\right\} \tag{10}$$

Equation (3) reduces to

$$2\frac{\partial F}{\partial \tau}-4\left(\frac{\Omega}{n}\right)^2F\frac{\partial F}{\partial y}=\frac{\partial P}{\partial y}+\frac{2}{R}\frac{\partial^2 F}{\partial y^2} \tag{11}$$

and serves merely to determine the axial pressure gradient.

Transverse component

On the assumption that the amplitude of the oscillations, i.e. Ω/n, is small, it appears permissible to neglect to first-order the non-linear convection terms in (7), (8) and (11), though retaining the non-linear centrifugal term $(ge^{i\tau})^2$. (Some remarks concerning the validity of this assumption will be made later.) For the transverse flow we then have to solve

$$\frac{d^2 g}{dy^2}-iRg=0$$

with $g(0)=1$ and $g(1)=0$. The solution is

$$g=\frac{\sinh\sqrt{(iR)}\,(1-y)}{\sinh\sqrt{(iR)}}. \tag{12}$$

Equation (12) leads to, in real notation,

$$\frac{v}{r\Omega}=\frac{\begin{aligned}&[\cos\sqrt{(\tfrac{1}{2}R)}\,y\cosh\sqrt{(\tfrac{1}{2}R)}\,(2-y)-\cos\sqrt{(\tfrac{1}{2}R)}\,(2-y)\cosh\sqrt{(\tfrac{1}{2}R)}\,y]\cos nt\\ &+[\sin\sqrt{(\tfrac{1}{2}R)}\,y\sinh\sqrt{(\tfrac{1}{2}R)}\,(2-y)-\sin\sqrt{(\tfrac{1}{2}R)}\,(2-y)\sinh\sqrt{(\tfrac{1}{2}R)}\,y]\sin nt\end{aligned}}{\cosh\sqrt{(2R)}-\cos\sqrt{(2R)}}. \tag{13}$$

Expanding this for small values of the Reynolds number, we get

$$\frac{v}{r\Omega}=(1-y)\left\{\left[1-\frac{R^2}{360}y(8+8y-12y^2+3y^3)\right]\cos nt+\frac{R}{6}y\,(2-y)\sin nt\right\}+O(R^3), \tag{14}$$

with amplitude approximately

$$\frac{|v|}{r\Omega}\approx(1-y)\left[1-\frac{R^2}{180}y(2-y)\,(2-2y+y^2)\right] \tag{15}$$

and phase angle

$$\tan^{-1}\tfrac{1}{6}Ry(2-y). \tag{16}$$

For large R, (13) becomes

$$\frac{v}{r\Omega} \sim e^{-\sqrt{(\frac{1}{2}R)}y}\cos\left(nt-\sqrt{(\tfrac{1}{2}R)}\,y\right), \tag{17}$$

the well-known shear-layer solution, whose magnitude is

$$\frac{|v|}{r\Omega} = e^{-\sqrt{(\frac{1}{2}R)}y}. \tag{18}$$

Numerical evaluation of $|v|/r\Omega$ from (13), (15) and (18) for various values of R indicates that the approximation (15) is valid for $R < 10$, and the asymptotic form (18) for $R > 20$. Curves illustrating $|v|/r\Omega$ for $R = 5$, 50 and 200 are given in figure 1, the first derived from (15), the two latter from (18).

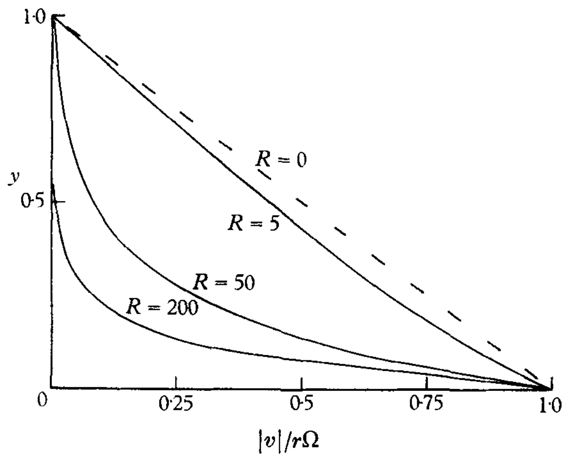

FIGURE 1. One disk oscillating: amplitude of transverse velocity $|v|/r\Omega$ for $R = 5$, 50, 200.

We see that for small Reynolds numbers the fluid has transverse motion for all $0 \leqslant y \leqslant 1$. In the limit $R \to 0$, the velocity profile is linear in y (the broken line in figure 1) with zero-phase angle. For non-zero R, the profile assumes the form of a polynomial in y, and the fluid acquires a phase lag with respect to the oscillations of the plate. The curve $R = 5$ shows that $|v|/r\Omega$ decreases with increasing R; at the same time this decrease is less for small y than for large y. That is, there is already a trend towards concentration of the moving fluid near the oscillating plate.

At about $R = 50$, a boundary layer adjacent to the rotating disk becomes apparent, and its thickness has order of magnitude

$$y \sim \frac{1}{\sqrt{R}} = \frac{1}{d}\sqrt{\frac{\nu}{n}} \quad \text{or} \quad z \sim \frac{d}{\sqrt{R}} = \sqrt{\frac{\nu}{n}}. \tag{19}$$

Thus for large R the presence of the stationary disk has no effect on the transverse motion whose velocity profile is exactly that in the case of a single disk in an otherwise unbounded fluid.

From equation (14), the skin friction for small R on the disk $z = 0$ is

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=0} \approx -\frac{\mu r\Omega}{d}\left[\left(1+\frac{R^2}{45}\right)\cos nt - \frac{R}{3}\sin nt\right] \tag{20}$$

and on the disk $z = d$ is

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=d} \approx -\frac{\mu r\Omega}{d}\left[\left(1-\frac{7R^2}{360}\right)\cos nt + \frac{R}{6}\sin nt\right]. \tag{21}$$

When $R \to 0$, the shearing forces are equal on the two disks. As R increases, the shear on the rotating disk increases in magnitude, while that on the stationary disk decreases.

Similarly for large R, from equation (17)

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=0} \sim -\frac{\mu r\Omega}{d}\sqrt{\frac{R}{2}}\,(\cos nt - \sin nt) \tag{22}$$

and

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=d} \sim -\frac{\mu r\Omega}{d}\sqrt{(2R)}\,e^{-\sqrt{(\frac{1}{2}R)}} \simeq 0. \tag{23}$$

Steady radial-axial component

Reverting to equation (7), we see that the radial-axial flow will be given by

$$\frac{\partial^2 F}{\partial y\,\partial \tau} - (g\,e^{i\tau})^2 = -K(\tau) + \frac{1}{R}\frac{\partial^3 F}{\partial y^3}, \tag{24}$$

provided the non-linear convective terms can again be neglected. From (12) we obtain

$$(g\,e^{i\tau})^2 = \frac{1}{2}\left[\frac{\cosh\lambda y_1 - \cos\lambda y_1}{\cosh\lambda - \cos\lambda}\right] + \frac{1}{2}\left[\frac{\cosh(1+i)\lambda y_1 - 1}{\cosh(1+i)\lambda - 1}\right]e^{2i\tau}, \tag{25}$$

where $\lambda \equiv \sqrt{(2R)}, \quad y_1 = 1-y.$

This suggests that there are solutions of the form

$$F(y,\tau) = f(y) + h(y)\,e^{2i\tau} \tag{26}$$

and

$$K(\tau) = K_0 + K_1\,e^{2i\tau}. \tag{27}$$

Substitution of (25)–(27) into (24) yields

$$\frac{2}{\lambda^2}\frac{d^3 f}{dy^3} = K_0 - \frac{1}{2}\left[\frac{\cosh\lambda y_1 - \cos\lambda y_1}{\cosh\lambda - \cos\lambda}\right] \tag{28}$$

and

$$\frac{2}{\lambda^2}\frac{d^3 h}{dy^3} - 2i\frac{dh}{dy} = K_1 - \frac{1}{2}\left[\frac{\cosh(1+i)\lambda y_1 - 1}{\cosh(1+i)\lambda - 1}\right]. \tag{29}$$

The boundary conditions are

$$\left.\begin{aligned} f = f' = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = 1,\\ h = h' = 0 \quad \text{on} \quad y = 0 \quad \text{and} \quad y = 1.\end{aligned}\right\} \tag{30}$$

The above equations indicate that the radial-axial flow has a mean steady component and a fluctuating component of frequency twice that of the oscillating plate. The radial pressure gradient is similarly compounded of two such terms.

The solution of (28) for the mean steady component is found to be

$$f(y) = \frac{1}{(\cosh\lambda - \cos\lambda)}\left[\frac{1}{4\lambda}(\sinh\lambda y_1 + \sin\lambda y_1) - \frac{1}{4\lambda}(\sinh\lambda + \sin\lambda)(1-3y^2+2y^3)\right.$$
$$\left. + \tfrac{1}{4}(\cosh\lambda + \cos\lambda)\,y(1-2y+y^2) - \tfrac{1}{2}y^2(1-y)\right], \quad (31)$$

$$f'(y) = \frac{1}{(\cosh\lambda - \cos\lambda)}\left[-\tfrac{1}{4}(\cosh\lambda y_1 + \cos\lambda y_1) + \frac{3}{2\lambda}(\sinh\lambda + \sin\lambda)\,y(1-y)\right.$$
$$\left. + \tfrac{1}{4}(\cosh\lambda + \cos\lambda)(1-4y+3y^2) - \tfrac{1}{2}y(2-3y)\right] \quad (32)$$

with $$K_0 = \frac{3}{\lambda^3}\left[\frac{2\lambda + \lambda(\cosh\lambda + \cos\lambda) - 2(\sinh\lambda + \sin\lambda)}{\cosh\lambda - \cos\lambda}\right]. \quad (33)$$

With λ and y_1 replaced by $\sqrt{(2R)}$ and $1-y$, these expressions reduce to, for small Reynolds numbers,

$$f \approx \frac{R}{120}y^2(1-y)^2(3-y) + O(R^3), \quad (34)$$

$$f' \approx \frac{R}{120}y(1-y)(6-15y+5y^2) + O(R^3) \quad (35)$$

and $$K_0 \approx \tfrac{3}{20} + O(R^2). \quad (36)$$

For large R the corresponding approximations are

$$f \sim \tfrac{1}{4}y(1-y)^2 - \frac{1}{4\sqrt{(2R)}}(1-y)^2(1+2y) + \frac{1}{4\sqrt{(2R)}}e^{-\sqrt{(2R)}y}, \quad (37)$$

$$f' \sim \tfrac{1}{4}(1-y)(1-3y) + \frac{3}{2\sqrt{(2R)}}y(1-y)\,\tfrac{1}{4}e^{-\sqrt{(2R)}y} \quad (38)$$

and $$K_0 \sim \frac{3}{R}\left(\frac{1}{2} - \frac{1}{\sqrt{(2R)}}\right). \quad (39)$$

These solutions are illustrated in figures 2 and 3. Figure 2 depicts schematically a typical streamline of the radial-axial steady flow at each of the selected Reynolds numbers $R = 5$, 50, 200; figure 3 shows the dimensionless radial velocity $f' \equiv nu/r\Omega^2$. When $R = 5$, $f' = 0$ at $y = 0{\cdot}47$ and the curves for $y < 0{\cdot}47$ and $y > 0{\cdot}47$ have almost the same shape. As R increases, however, there is a marked departure from this symmetry. When $R = 200$ we see that u is positive for $y < 0{\cdot}38$, where both centrifugal force and pressure gradient are active. In this region the exponential term in (38) is of the same order of magnitude as the first term and the profile is of the boundary-layer type. For $y > 0{\cdot}38$ it is seen from figure 1 that the centrifugal force is vanishingly small, and the flow near the stationary disk behaves essentially like Poiseuille flow with the pressure gradient (39).

It is important to note here that a solution for the steady radial-axial flow has been obtainable in which inertia terms are neglected, and yet all the boundary conditions are satisfied. Such a solution could not be found in the single-disk problem (Rosenblat 1959). An examination of the behaviour at high Reynolds numbers throws light on the contrasting features of the two problems.

In both cases centrifugal force, acting within a boundary layer of thickness $\sqrt{(\nu/n)}$, causes fluid to be thrown radially outwards. As compensation, fluid is drawn inwards along the axis towards the moving disk. In the single-disk case, the quantity of fluid required for this purpose can be obtained in a purely axial motion from, in effect, a source at infinity whose strength is determined by continuity considerations. Hence no radial flow takes place at large z, and there

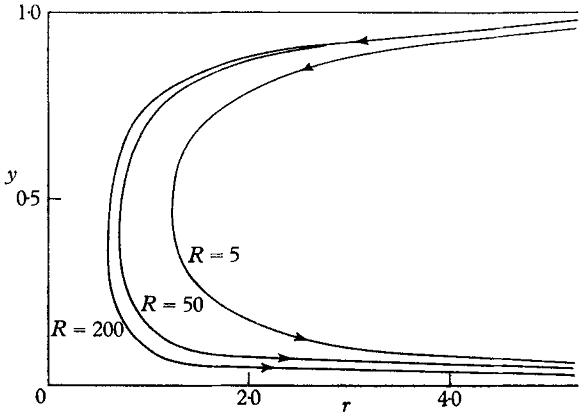

FIGURE 2. One disk oscillating: typical streamlines of steady radial-axial flow for $R = 5, 50, 200$.

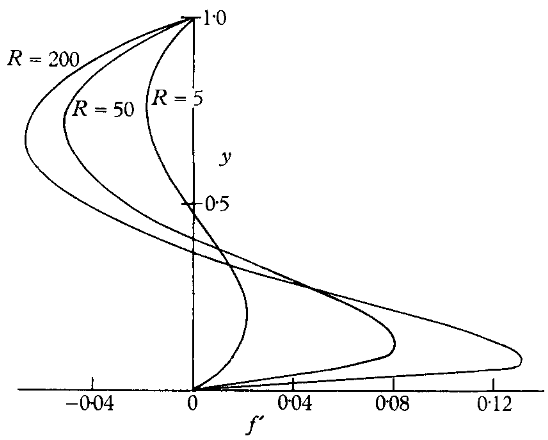

FIGURE 3. One disk oscillating: steady radial velocity $f' = nu_s/r\Omega^2$ for $R = 5, 50, 200$.

is no radial pressure gradient. When a second disk is present, however, the condition $w = 0$ on $z = d$ has to be satisfied. Consequently the fluid thrown radially outwards near the oscillating disk must be balanced by fluid sucked radially inwards. To maintain this inward flow, a radial pressure gradient is induced.

Again in the single-disk problem the centrifugal force is zero outside the boundary layer. The hitherto-neglected inertia terms become comparable in magnitude with the viscous term and hence important. The result is the formation of a second boundary layer within which the steady radial flow takes place. In the present problem, on the other hand, although outside the boundary layer the non-linear inertia terms are again large compared with the vanishingly small centrifugal force, they are still negligible in relation to the radial pressure gradient which now dominates (in general, see later). Hence the boundary conditions are satisfied and there is no second layer formed.

Fluctuating radial-axial flow

Equation (29) yields the time-dependent component of the radial-axial flow. For low Reynolds numbers the solution is found to be

$$h = \frac{R}{120}y^2(1-y)^2\left[(3-y)-\frac{iR}{120}(29+167y-150y^2+30y^3)\right]+O(R^3), \tag{40}$$

$$h' = \frac{R}{120}y(1-y)\times\left[(6-15y+5y^2)-\frac{iR}{120}(58+385y-143y^2+1050y^3-210y^4)\right]+O(R^3) \tag{41}$$

and

$$K_1 = \frac{3}{20}-\frac{109iR}{4200}+O(R^2). \tag{42}$$

For high R we obtain

$$8i(2-\surd(2iR))\,h \sim \frac{(\surd 2-2)(\surd(2iR)-1)}{\surd(2iR)}-(\surd 2-2)\,y+\frac{(2-\surd(2iR))}{\surd(iR)}e^{-2\surd(iR)y} + \frac{(2\surd(2iR)-2-\surd 2)}{\surd(2iR)}e^{-\surd(2iR)y}+\frac{(\surd 2-2)}{\surd(2iR)}e^{-\surd(2iR)(1-y)}, \tag{43}$$

$$8i(2-\surd(2iR))\,h' \sim 2-\surd 2-2(2-\surd(2iR))\,e^{-2\surd(iR)y}-(2\surd(2iR)-2-\surd 2)\,e^{-\surd(2iR)y} + (\surd 2-2)\,e^{-\surd(2iR)(1-y)} \tag{44}$$

and

$$K_1 \sim \frac{2-\surd 2}{4(\surd(2iR)-2)}. \tag{45}$$

The principal feature of this component of the flow is that at high Reynolds numbers a boundary layer forms near the stationary disk as well as near the rotating disk—in contrast to the steady component. This arises from the last term in equation (44). When y is nearly 1, (44) reduces to

$$h' \sim \frac{(\surd 2-2)}{8i(\surd(2iR)-2)}[1-e^{-\surd(2iR)(1-y)}],$$

a typical boundary-layer profile, which is very similar to the solution for flow in a pipe under a periodic pressure gradient (Schlichting 1955, p. 199). In both cases the formation of a boundary layer is due to the fact that the vorticity diffusing away from the wall is constantly changing in sign.

Outside the two layers the velocity—given then by the first term of (44)—is independent of y, since only the constant pressure gradient is there active.

Remarks

Finally, we examine the permissibility of neglecting the inertia terms of equations (7) and (8), with particular reference to flow at high Reynolds numbers. From (38) and (39) it is seen that for large $R, f' = O(1)$ and $K_0 = O(1/R)$. Therefore the term $(\Omega/n)^2 f'^2$ which occurs in equation (7) would appear to be small compared with the linear terms if

$$(\Omega/n)^2 \ll 1/R.$$

Since $1/\surd R = (1/d)\surd(\nu/n)$, and Ω/n is the amplitude of the disk's oscillations, we can say that equation (28) for f and f' is a valid approximation provided that

$$\left.\begin{aligned} \text{amplitude of oscillations} &\ll \frac{\text{boundary-layer thickness}}{\text{distance between disks}}, \\ \frac{\Omega}{n} &\ll \sqrt{\frac{\nu}{n}}\Big/ d. \end{aligned}\right\} \tag{46}$$

An alternative way of expressing this condition is as follows. Since by equation (6) the steady radial component is $u_s = (r\Omega^2/n)f'$, the rate of convection by the radial flow through a typical distance r is Ω^2/n. Also the rate at which vorticity diffuses through a distance d is ν/d^2. Hence (46) is equivalent to

$$\left.\begin{aligned} &\text{rate of convection by } u_s \ll \text{rate of diffusion through distance } d, \\ \text{i.e.}\quad &\Omega^2/n \ll \nu/d^2. \end{aligned}\right\} \tag{47}$$

The right-hand side of (47) represents the influence of the second disk. If this is absent, $d \to \infty$, $\nu/d^2 \to 0$ so that convection is always significant—analogous to the result obtained in the single-disk problem. In the present case inertia need be included only if (46) or (47) is not satisfied. We would then have to solve a complicated non-linear equation very similar to that encountered by Batchelor (1951) and Stewartson (1953) for the steadily rotating disks.

These difficulties arise only for the steady radial-axial component of velocity. For it is quickly seen that linearization of the equations determining the transverse and fluctuating radial components does not require such a stringent condition. In these cases it is in fact only necessary that the amplitude Ω/n be small compared with unity.

3. Both disks oscillating

We now consider the case where the two disks are oscillating with the same frequency and angular speed but 180° out of phase. This is analogous to the problem, studied by Stewartson, of disks rotating steadily with the same speeds in opposite directions. As in that problem we anticipate that the results will be symmetrical (or anti-symmetrical) about the plane $y = \frac{1}{2}$.

The boundary conditions (5) are replaced by

$$\left.\begin{aligned} u = w = 0, \quad v &= r\Omega e^{int} \quad \text{on} \quad z = 0, \\ u = w = 0, \quad v &= -r\Omega e^{int} \quad \text{on} \quad z = d. \end{aligned}\right\} \tag{5a}$$

Equations (1)–(4) may again be transformed using (6) and for the transverse flow lead to

$$\frac{d^2g}{dy^2} - iRg = 0,$$

as previously, but with $g(0) = 1$ and $g(1) = -1$. The solution now is

$$g = \frac{\sinh\surd(iR)(1-y) - \sinh\surd(iR)\,y}{\sinh\surd(iR)}. \tag{12a}$$

For small Reynolds numbers this yields a velocity profile

$$\frac{v}{r\Omega} = (1-2y)\left\{\left[1-\frac{R^2}{360}y(1-y)(1+3y-3y^2)\right]\cos nt+\frac{R}{6}y(1-y)\sin nt\right\}+O(R^3). \quad (14a)$$

Hence the amplitude is given by

$$\frac{|v|}{r\Omega} = (1-2y)\left[1-\frac{R^2}{360}y(1-y)(1-2y+2y^2)\right] \quad (15a)$$

and the phase lag is $\tan^{-1}\frac{1}{6}Ry(1-y)$. (16a)

The skin friction is given by

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=d} = -\frac{2\mu r\Omega}{d}\left[\left(1+\frac{R^2}{720}\right)\cos nt-\frac{R}{12}\sin nt\right]. \quad (20, 21a)$$

These approximations are found to be valid for about $R < 40$. The amplitude profile for $R = 20$, illustrated in figure 4, is antisymmetrical about the plane

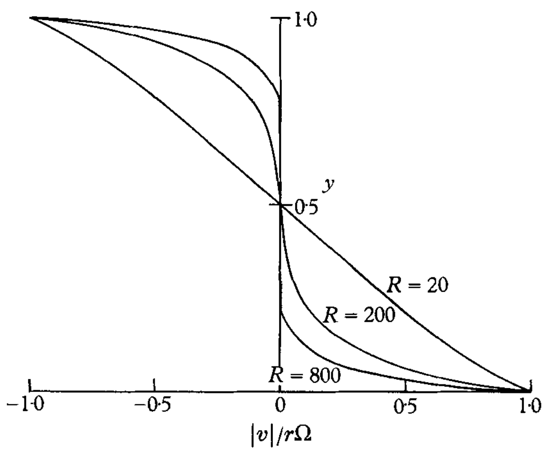

FIGURE 4. Two disks oscillating: amplitude of transverse velocity $|v|/r\Omega$ for $R = 20, 200, 800$.

$y = \frac{1}{2}$ in which there is zero transverse flow. The shearing forces on the two disks are equal in magnitude, though they will be opposite in sign.

The asymptotic solution, useful *for* $R > 80$, is

$$\frac{v}{r\Omega} \sim e^{-\sqrt{(\frac{1}{2}R)}y}\cos[nt-\sqrt{(\tfrac{1}{2}R)}\,y]-e^{-\sqrt{\frac{1}{2}(R)}(1-y)}\cos[nt-\sqrt{(\tfrac{1}{2}R)}\,(1-y)], \quad (17a)$$

and this leads to

$$\mu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \mu\left(\frac{\partial v}{\partial z}\right)_{z=d} \sim -\frac{\mu r\Omega}{d}\sqrt{(\tfrac{1}{2}R)}\,(\cos nt-\sin nt). \quad (22, 23a)$$

Figure 4 shows also the velocity amplitude for $R = 200, 800$ derived from (17a). It is seen that a boundary layer of thickness $z \sim \sqrt{(\nu/n)}$ forms adjacent to both of the disks. The expression for the skin friction is identical with (22), when one disk is stationary.

For the radial-axial flow equation (24) again holds, with the same boundary conditions. Neglect of the inertia terms is allowable here also provided the condition (46) is satisfied. The substitutions (26) and (27) yield two equations analogous to (28) and (29), with a difference in the last term in each owing to the changed value of $(g\,e^{i\tau})^2$.

After some tedious algebra we obtain the following results for the mean steady component of the flow. For small R,

$$f = \frac{R}{60}y^2(1-y)^2(1-2y)+O(R^3), \tag{34a}$$

$$f' = \frac{R}{30}y(1-y)(1-5y+5y^2)+O(R^3) \tag{35a}$$

and

$$K_0 = \tfrac{1}{10}+O(R^2). \tag{36a}$$

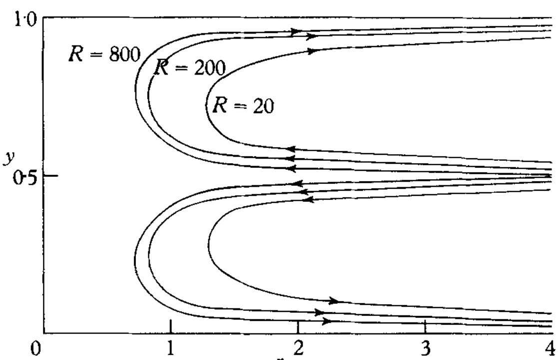

FIGURE 5. Two disks oscillating: typical streamlines of steady radial-axial flow for $R = 20, 200, 800$.

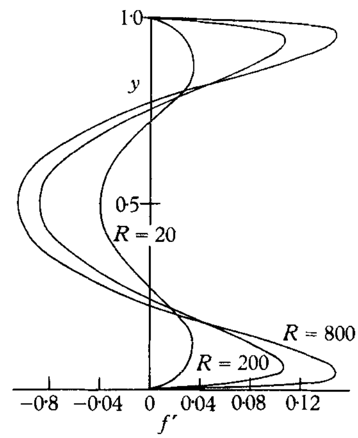

FIGURE 6. Two disks oscillating: steady radial velocity $f' = nu_s/r\Omega^2$ for $R = 20, 200, 800$.

For large Reynolds numbers,

$$f \sim \tfrac{1}{4}y(1-y)(1-2y)-\frac{1}{4\sqrt{(2R)}}(1-2y)(1+2y-2y^2)+\frac{1}{4\sqrt{(2R)}}[e^{-\sqrt{(2R)}y}-e^{-\sqrt{(2R)}(1-y)}], \tag{37a}$$

$$f' \sim \tfrac{1}{4}(1-6y+6y^2)+\frac{3}{\sqrt{(2R)}}y(1-y)-\tfrac{1}{4}[e^{-\sqrt{(2R)}y}+e^{-\sqrt{(2R)}(1-y)}] \tag{38a}$$

and

$$K_0 \sim \frac{3}{R}\left(1-\frac{2}{\sqrt{(2R)}}\right). \tag{39a}$$

Figure 5 depicts representative streamlines of this flow at $R = 20, 200, 800$. As might be expected there is symmetry about the plane $y = \frac{1}{2}$ and no axial flow across it. In this sense this plane acts as an interface separating the fluid into two self-contained portions, in each of which there is axial flow towards the moving disk. The pressure gradient at high R given by (39a) is twice that of § 2, as it has to counterbalance the flow in two boundary layers. In both cases, however, this pressure gradient is quite small at large Reynolds numbers, which corresponds to a result obtained by Stewartson.

In figure 6 we see the radial velocity component at each of the three values of the Reynolds number. In all cases f' is a minimum (i.e. maximum inflow) when

$y = \frac{1}{2}$. For $R = 20$ it is zero on planes $y = 0{\cdot}27$ and $0{\cdot}73$. Between these and the walls there is radial outflow, due to the predominance of centrifugal action. For $0{\cdot}27 < y < 0{\cdot}73$ there is an inflow as the pressure gradient is then the main factor.

At high Reynolds numbers the profile has the character of a boundary layer near both disks, for about $y < 0{\cdot}2$ and $y > 0{\cdot}8$. In the main body of the fluid its shape is described by a quadratic in y, which is well known as the behaviour of a fluid under constant pressure gradient.

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